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On the Anisotropic Walk on the Supercritical Percolation Cluster

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Abstract: We investigate in this work the asymptotic behavior of an anisotropic random walk on the supercritical cluster for bond percolation on $\mathbb{Z}^d$, $d \geq 2$. In particular we show that for small anisotropy the walk behaves in a ballistic fashion, whereas for strong anisotropy the walk is sub-diffusive. For arbitrary anisotropy, we also prove the directional transience of the walk and construct a renewal structure.

0. Introduction

We investigate here the asymptotic behavior of an anisotropic random walk on the supercritical infinite cluster of $\mathbb{Z}^d$ for bond percolation. Much work has been devoted to the study of a random walk on the supercritical cluster, when at each step the particle jumps with equal probability to one of the neighboring sites on the cluster, see for instance [3, 15, 19–21]. Much less is known in the anisotropic situation, where some preferred direction of jump is present. Conjectures on the behavior of such a walk can be found in the theoretical physics literature, see for instance Havlin-Bunde [16], pp. 136–139. In contrast to the naive intuition, these conjectures predict a phase transition from a ballistic to a sub-ballistic behavior, as the bias of the walk increases. One object of the present work is to investigate this effect.

Before returning to this question, we first describe the model more precisely. We let B_d stand for the set of nearest neighbor bonds (or edges) on $\mathbb{Z}^d$, $d \geq 2$, and $\Omega = \{0, 1\}^{B_d}$ for the set of configurations. Thus a bond $b \in B_d$ is open (resp. closed) in the configuration $\omega \in \Omega$, when $\omega(b) = 1$, (resp. $\omega(b) = 0$), and ω naturally induces a partition of $\mathbb{Z}^d$ into open clusters (or connected components). We denote by $\mathbb{P}$ the product measure on Ω endowed with its canonical σ-algebra, under which the canonical coordinates are Bernoulli variables with success probability $p \in (0, 1)$. The anisotropic walk with direction $\widehat{\ell} \in S^{d-1}$ and strength $\lambda > 0$ in the configuration ω is then the Markov chain on $\mathbb{Z}^d$, with transition probability $r_\omega(x, y)$, $x, y \in \mathbb{Z}^d$, such that:

$$\begin{aligned} r_\omega(x,x) &= 1, \quad \text{if all edges incident to } x \text{ are closed},\\ r_\omega(x,y) &= \frac{e^{\ell\cdot(y-x)}}{n_\omega(x)}, \quad \text{if } y \sim x, \text{ (i.e. } y \text{ is a neighbor of } x\text{), and } \omega(\{x,y\}) = 1,\\ r_\omega(x,y) &= 0, \quad \text{in all other cases,} \end{aligned} \tag{0.1}$$

where we have used the notations

$$\ell = \lambda\,\widehat{\ell}, \text{ and} \tag{0.2}$$

$$\begin{aligned} n_\omega(x) = &\sum_{z\sim x,\omega(\{x,z\})=1} e^{\ell\cdot(z-x)}, \text{ if some edge incident to } x \text{ is open},\\ &= 1, \text{ otherwise}. \end{aligned} \tag{0.3}$$

We denote by $P_{x,\omega}$ the canonical law on $(\mathbb{Z}^d)^{\mathbb{N}}$ of the above Markov chain starting at $x \in \mathbb{Z}^d$, and by $(X_n)_{n\ge 0}$, the canonical process. It is plain that the open clusters are the irreducible components of this Markov chain, that admits the reversible measure

$$m_\omega(x) = e^{2\ell\cdot x} n_\omega(x),\ x \in \mathbb{Z}^d, \text{ i.e.} \tag{0.4}$$

$$m_\omega(x)\, r_\omega(x,y) = m_\omega(y)\, r_\omega(y,x),\ x, y \in \mathbb{Z}^d, \omega \in \Omega\,. \tag{0.5}$$

It is also convenient to introduce the semi-direct product measure

$$P_x = \mathbb{P} \times P_{x,\omega},\ x \in \mathbb{Z}^d\,. \tag{0.6}$$

If $p_c(d) \in (0,1)$, denotes the critical probability for bond percolation on $\mathbb{Z}^d$, cf. [14], we are chiefly interested in the supercritical regime when

$$p > p_c(d)\,, \tag{0.7}$$

so that $\mathbb{P}$-a.s., ω induces a unique infinite cluster $\mathcal{C}$, see [14], p. 98. Our main purpose is the investigation of the asymptotic behavior of the walk on $\mathcal{C}$, and we consider the event of positive $\mathbb{P}$-probability:

$$\mathcal{I} = \text{there is an infinite open cluster which is unique and contains 0}, \tag{0.8}$$

as well as the conditioned measure:

$$P = P_0[\cdot\,|\,\mathcal{I}]\,. \tag{0.9}$$

We now turn to the description of the main results of this article. We show in Theorem 1.2, that the walk is always transient in the direction $\widehat{\ell}$, that is:

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}[\lim_n X_n \cdot \widehat{\ell} = \infty] = 1\,. \tag{0.10}$$

The proof of (0.10) relies on certain energy estimates. With the help of (0.10), we define regeneration times for the walk under the measure P. This is somewhat in the spirit of Shen [23], or Sznitman-Zerner [28]. There is however a special twist due to the conditioning present in (0.9), and in contrast to [23, 28], our regeneration times are configuration dependent. In this fashion we obtain a renewal structure for the walk under P, regardless of the strength and direction of the anisotropy, see Theorem 2.4. Such a renewal structure is a powerful tool as the recent developments on random walks in random environment clearly demonstrate, cf. [26–29].

Let us then discuss the influence of the strength of the anisotropy. For weak anisotropy we show that the walk behaves in a ballistic fashion. Namely for $0 < \lambda < \lambda_w(d, p)$, we prove in Theorem 3.4 both a law of large numbers:

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s., } \lim_n \frac{X_n}{n} = v\,, \tag{0.11}$$

with v deterministic and $v \cdot \ell > 0$, and a functional central limit theorem governing the correction to the law of large numbers. Denoting by $D(\mathbb{R}_+, \mathbb{R}^d)$ the space of right continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ with left limits, endowed with the Skorohod topology, cf. [13], we prove that under P,

$$\begin{aligned} &B^n_\cdot = \frac{1}{\sqrt{n}}\,(X_{[\cdot n]} - [\cdot n]v) \text{ converges in law on } D(\mathbb{R}_+, \mathbb{R}^d) \text{ to a} \\ &\text{Brownian motion with non-degenerate covariance matrix.} \end{aligned} \tag{0.12}$$

The main ingredients in the proof of (0.11), (0.12), are the above mentioned renewal structure, as well as upper bounds on the probability of occurrence of low principal Dirichlet eigenvalues for the walk on $\mathcal{C}$ killed when exiting a large box, and controls on the exit distribution from a large box under P, see Lemma 3.1, 3.2.

As alluded to above, an interesting feature of the model is that strengthening the anisotropy does not speed up the walk. We show in Theorem 4.1 that for $\lambda > \lambda_s(d, p)$,

$$\mathbb{P}\text{-a.s. for } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s., } \lim_n \frac{X_n}{n^{\lambda_s/\lambda}} = 0\,. \tag{0.13}$$

Thus strong anisotropy leads to sub-ballistic and even sub-diffusive behavior. Intuitively, the effect is due to the presence of long dangling ends on the infinite cluster, which depending on their general direction can turn into powerful traps where the walk tends to spend a very long time, leading to a massive slowdown of the walk. This scenario can be found in the theoretical physics literature, see [16] or Dhar-Stauffer [11] and references therein. This has a similar flavor to what is known to happen for certain one-dimensional random walks in random environment, cf. Solomon [25], Kesten-Kozlov-Spitzer [17], Sinai [24], or for the random craters models of Bramson-Durrett [7], Bramson [6], or for some random walks on inhomogeneous trees, cf. Lyons-Pemantle-Peres [18], or for some models related to the investigation of the aging phenomenon, see [5].

Let us then describe the structure of the present article. In Sect. 1, we introduce further notations and develop the necessary energy estimates that enable to prove transience in the direction $\widehat{\ell}$ for arbitrary non-vanishing anisotropy, see Theorem 1.2.

In Sect. 2, we define certain regeneration times, see (2.13), (2.23), and prove the key renewal property in Theorem 2.4. We also dominate the displacement of the walk in the direction $\widehat{\ell}$ at the first regeneration time, in Proposition 2.5. This control is later used in the analysis of the walk in the weak anisotropy regime.

Section 3 discusses the ballistic nature of the walk in the weak anisotropy regime. The law of large numbers and the central limit theorem appear in Theorem 3.4. The necessary bounds on the occurrence of low principal Dirichlet eigenvalues and on the exit distribution from a large box are provided in Lemmas 3.1, 3.2.

Section 4 studies the strong anisotropy regime. The main result showing the sub-diffusive nature of the walk for large λ can be found in Theorem 4.1.

Finally let us explain the convention used concerning constants. Throughout the text we denote by c a positive constant depending only on d and p, with value changing from place to place. The dependence on additional parameters is otherwise mentioned in the notation, so that for instance $c(\lambda)$ denotes a positive constant depending on d, p, λ.

After finishing this article we learnt of an independent work [4], by N. Berger, N. Gantert and Y. Peres, which at the time was in the process of being completed. This article studies a random walk on the two-dimensional supercritical cluster, that is biased along the first coordinate axis. In this context the authors prove the directional transience of the walk corresponding to (0.10); they also show that for small bias the walk has a non-vanishing velocity, and that for large enough bias the walk has null limiting velocity. Their proofs use a strategy quite distinct from the one followed here.

1. Directional Transience

In this section we introduce further notations and then prove with the help of energy estimates that the walk on the infinite cluster is always transient in the direction $\widehat{\ell}$, cf. (0.10) or Theorem 1.2. We tacitly assume (0.7) throughout the article.

We begin with some additional notations and recall some classical facts. We denote by $|\cdot|$ the Euclidean distance on $\mathbb{R}^d$ and by $(e_i)_{1\le i\le d}$ the canonical basis of $\mathbb{R}^d$. For U a subset of $\mathbb{Z}^d$, $|U|$ stands for the cardinality of U and ∂U for the boundary of U:

$$\partial U = \{x \in \mathbb{Z}^d \backslash U,\ \exists y \in U,\ |x-y| = 1\}. \tag{1.1}$$

For $x \in \mathbb{Z}^d$ we will sometimes consider the discrete half-spaces:

$$\mathcal{H}_x^+ = \{x \in \mathbb{Z}^d,\ z\cdot\ell \ge x\cdot\ell\},\ \mathcal{H}_x^- = \{z \in \mathbb{Z}^d,\ z\cdot\ell \le x\cdot\ell\}. \tag{1.2}$$

For a bond $b = \{x, y\} \in B_d$, and $z \in \mathbb{Z}^d$, we write $b + z$ for $\{x+z, y+z\} \in B_d$, and denote by $(t_z)_{z\in\mathbb{Z}^d}$ the spatial shift on Ω. From time to time we will consider the measurable set with full $\mathbb{P}$-measure:

$$\Omega_1 = \{\omega \in \Omega, \text{there is an infinite open cluster and it is unique}\}, \tag{1.3}$$

and for $\omega \in \Omega_1$, we will denote by $\mathcal{C}$ the infinite open cluster. For $C \subseteq B_d$, $\mathcal{O}_C$ stands for the event:

$$\mathcal{O}_C = \{\omega \in \Omega,\ \omega(b) = 1, \text{for all } b \in C\}. \tag{1.4}$$

Given $\omega \in \Omega$, an open path will stand for a nearest neighbor path on $\mathbb{Z}^d$, for which each step corresponds to an open bond of ω. Since $p > p_c(d)$, percolation occurs on $\mathbb{Z}_+^d$ as well, see for instance [14], pp. 148, 304, and the following events have positive $\mathbb{P}$-measure:

$$\begin{aligned}\mathcal{I}^\pm = \{\omega \in \Omega, &\ 0 \text{ belongs to an infinite open cluster of } \mathcal{H}_0^\pm \text{ induced}\\ &\text{by the restriction of } \omega \text{ to edges between vertices in } \mathcal{H}_0^\pm\},\end{aligned} \tag{1.5}$$

$$\mathcal{I}_x^\pm = (t_x)^{-1}(\mathcal{I}^\pm),\ \mathcal{I}_x = (t_x)^{-1}\,(\mathcal{I}),\ x \in \mathbb{Z}^d, \text{(cf. (0.8) for the notation)}. \tag{1.6}$$

We let $(\theta_n)_{n\geq 0}$, and $(\mathcal{F}_n)_{n\geq 0}$, respectively stand for the canonical shift and filtration on $(\mathbb{Z}^d)^{\mathbb{N}}$. For $U \subseteq \mathbb{Z}^d$, H_U, $\widetilde{H}_U$, T_U respectively denote the entrance time, the hitting time, and the exit time of (X_n) in or from U:

$$\begin{aligned} H_U &= \inf\{n \geq 0,\ X_n \in U\},\ \ \widetilde{H}_U = \inf\{n \geq 1,\ X_n \in U\}, \\ T_U &= \inf\{n \geq 0,\ X_n \notin U\}. \end{aligned} \tag{1.7}$$

We will write H_x, $\widetilde{H}_x$ in place of $H_{\{x\}}$, $\widetilde{H}_{\{x\}}$, for $x \in \mathbb{Z}^d$.

The reversible character of the walk with transition probability (0.1) plays an important role. For $\omega \in \Omega$, $b = \{x, y\} \in B_d$, we define the weight of the edge b in the configuration ω:

$$w_\omega(b) = m_\omega(x)\, r_\omega(x, y) = m_\omega(y)\, r_\omega(y, x) = \exp\{\ell \cdot (x + y)\}\, \omega(b)\,. \tag{1.8}$$

When no confusion arises we drop the subscript ω and write $L^2(m)$ and $(\cdot, \cdot)_m$ for the Hilbert space of square m_ω-integrable functions on $\mathbb{Z}^d$, and its associated scalar product. We denote by R the transition kernel of the walk, so that for a function f on $\mathbb{Z}^d$:

$$R\, f(x) = \sum_y r_\omega(x, y)\, f(y),\ x \in \mathbb{Z}^d\,. \tag{1.9}$$

The Dirichlet form is then defined for $f, g \in L^2(m)$ via:

$$\mathcal{E}(f, g) = \big(f, (I - R)g\big)_m = \frac{1}{2} \sum_{|x-y|=1} \big(f(y) - f(x)\big)\, \big(g(y) - g(x)\big)\, w_\omega(\{x, y\})\,. \tag{1.10}$$

For U a non-empty subset of $\mathbb{Z}^d$, and $\omega \in \Omega$, the walk reflected in U will be the Markov chain with state space U and transition probability:

$$\begin{cases} r^U_\omega(x, x) = 1,\ \ x \in U, \text{ if for all } z \sim x,\ z \in U,\ \omega(\{x, z\}) = 0\,, \\ r^U_\omega(x, y) = \dfrac{w_\omega(\{x, y\})}{\sum\limits_{z\in U, z\sim x} w_\omega(\{x, z\})}\,, \ \text{ if } x, y \in U, x \sim y, \text{ and } \omega(\{x, y\}) = 1, \\ r^U_\omega(\cdot, \cdot) = 0,\ \ \text{otherwise}\,. \end{cases} \tag{1.11}$$

The measure on U:

$$m^U_\omega(x) = \sum_{y\in U} w_\omega(\{x, y\}),\ \ x \in U\,, \tag{1.12}$$

is then reversible for this Markov chain and the corresponding Dirichlet form (defined in analogy with (1.10)) is then:

$$\mathcal{E}^U(f, g) = \frac{1}{2} \sum_{\substack{x,y\in U \\ x\sim y}} \big(f(y) - f(x)\big)\, \big(g(y) - g(x)\big)\, w_\omega(\{x, y\}),\ \ f, g \in L^2(m^U_\omega)\,. \tag{1.13}$$

Incidentally note that when $U = \mathbb{Z}^d$, $r^U_\omega(\cdot, \cdot)$ coincides with r_ω in (0.1) but m^U_ω may differ from m_ω in (0.4), due to the possible presence of isolated sites.

We now turn to the proof of transience in the direction $\widehat{\ell}$. The crucial controls come from the next lemma. To state the lemma we still need to introduce the stopping times:

$$T_u = \inf\{n \geq 0,\ X_n \cdot \widehat{\ell} \geq u\},\ \ \widetilde{T}_u = \inf\{n \geq 0,\ X_n \cdot \widehat{\ell} \leq u\},\ u \in \mathbb{R}\,, \tag{1.14}$$

$$D_+ = \inf\{n \geq 1,\ X_n \cdot \widehat{\ell} \geq X_0 \cdot \widehat{\ell}\},\ D_- = \inf\{n \geq 1,\ X_n \cdot \widehat{\ell} \leq X_0 \cdot \widehat{\ell}\}\,. \tag{1.15}$$

Lemma 1.1.

$$P_0[D_+ > \widetilde{T}_{-L}] \leq e^{-c(\lambda)L},\ \ L > 1\,, \tag{1.16}$$

$$P_0\Big[\bigcap_{L>1} \{D_- > T_L\}\Big] \geq c(\lambda)\,, \tag{1.17}$$

(*see the end of the introduction for the convention used to denote positive constants*).

Proof. Both estimates will follow from energy considerations. We begin with (1.16). We choose

$$(f_i)_{1\leq i\leq d}, \text{ a basis of } \mathbb{R}^d, \text{ with } f_1 = \widehat{\ell}\,, \tag{1.18}$$

and for $L, \widetilde{L} > 1$, consider the discrete box:

$$U_0 = \{z \in \mathbb{Z}^d,\ -L < z \cdot f_1 < 0,\ \sup_{i\geq 2} |z \cdot f_i| < \widetilde{L}\}\,, \tag{1.19}$$

as well as the positive and negative parts of ∂U_0:

$$\partial_+ U_0 = \{z \in \partial U_0,\ z \cdot f_1 \geq 0\},\ \ \partial_- U_0 = \{z \in \partial U_0,\ z \cdot f_1 \leq -L\}\,. \tag{1.20}$$

For $\omega \in \Omega$, we will consider the walk reflected in U, see (1.11), where

$$U = U_0 \cup \partial U_0\,. \tag{1.21}$$

It follows from Dirichlet's principle, see (85) in Chapter 3 of Aldous-Fill [1], that

$$I_U(\omega) = \sum_{x\in\partial_+ U_0} m_\omega^U(x)\, P_{x,\omega}^U[H_{\partial_- U_0} < \widetilde{H}_{\partial_+ U_0}], \ \text{ with} \tag{1.22}$$

$$I_U(\omega) = \inf\{\mathcal{E}^U(f,f),\ f_{|\partial_+ U_0} = 1,\ f_{|\partial_- U_0} = 0\}\,, \tag{1.23}$$

and $P_{x,\omega}^U$ the canonical law of the walk reflected in U starting at x. Clearly $I_U(\cdot)$ increases when ω is replaced by $\overline{\omega}$ with

$$\overline{\omega}(b) = 1, \text{ for all } b \in B_d\,. \tag{1.24}$$

Thus integrating (1.22) over ω, we find:

$$\begin{aligned} I_U(\overline{\omega}) &\geq \sum_{\partial_+ U_0} \mathbb{E}\big[m_\omega^U(x)\, P_{x,\omega}^U[X_1 \in U_0,\ H_{\partial_- U_0}\circ\theta_1 < H_{\partial_+ U_0}\circ\theta_1]\big] \\ &\geq \sum_{\substack{x\in\partial_+ U_0, y\in U_0 \\ x\sim y}} e^{\ell\cdot(x+y)}\, \mathbb{E}\big[\omega(\{x,y\})\, P_{y,\omega}[H_{\partial_- U_0} = T_{U_0}]\big]\,, \end{aligned} \tag{1.25}$$

where we have used (1.8), (1.11), (1.12), the Markov property and the fact that the reflected walk and the walk coincide up to time T_{U_0}. Thus for $M > 0$, and $\sum'$ denoting the sum over $x \in \partial_+ U_0$, $y \in U_0$, $x \sim y$, with $\sup_{i \geq 2} |y \cdot f_i| < \widetilde{L} - M$, we see that:

$$I_U(\overline{\omega}) \geq e^{-\lambda} \sum{}' \mathbb{E}\big[\omega(\{x,y\})\, P_{y,\omega}[\widetilde{T}_{-L} < D_+, \sup_{i\geq 2, n \leq \widetilde{T}_{-L}} |(X_n - X_0)\cdot f_i)| < M]\big]$$
$$\geq c(\lambda) \sum{}' P_y[\widetilde{T}_{-L} < D_+, \sup_{i\geq 2, n \leq \widetilde{T}_{-L}} |(X_n - X_0)\cdot f_i| < M]\,. \tag{1.26}$$

Let us give some explanations on the last step. To this end for x, y, as above, we denote by E the event which appears under the $P_{y,\omega}$-probability in (1.26), and by ω_* the configuration which agrees with ω for all bonds different from $\{x, y\}$ and such that $\omega_*(\{x, y\}) = 0$. Note that when E occurs, the path up to time $\widetilde{T}_{-L}$ only touches $\{x, y\}$ at time 0. Thus for ω with $\omega(\{x, y\})\, P_{y,\omega}[E] > 0$, we find:

$$P_{y,\omega_*}[E] \leq P_{y,\omega}[E] \max_z \big(r_{\omega_*}(y,z)/r_\omega(y,z)\big)\,,$$

where the maximum runs over the (non-empty) collection of neighbors z of y with $z \cdot \widehat{\ell} < y \cdot \widehat{\ell}$ and $\omega(\{y, z\}) > 0$. From (0.1), this maximum is at most $1 + e^{2\lambda}$. Hence for ω as above we obtain

$$P_{y,\omega_*}[E] \leq P_{y,\omega}[E](1 + e^{2\lambda})\,,$$

and this inequality immediately extends to an arbitrary ω, since $P_{y,\omega}[E] = 0$ implies $P_{y,\omega_*}[E] = 0$. As a result, singling out the role of the variable $\omega(\{x, y\})$ in the $\mathbb{P}$-expectation, we see that

$$P_y[E] = \mathbb{E}\big[\omega(\{x,y\})\, P_{y,\omega}[E]\big] + \mathbb{E}[1\{\omega(\{x,y\}) = 0\}\, P_{y,\omega}[E]\big]$$
$$= \mathbb{E}\big[\omega(\{x,y\})\, P_{y,\omega}[E]\big] + \frac{1-p}{p}\, \mathbb{E}[\omega(\{x,y\})\, P_{y,\omega_*}[E]\big]$$
$$\leq \Big(1 + \frac{1-p}{p}(1 + e^{2\lambda})\Big)\, \mathbb{E}\big[\omega(\{x,y\})\, P_{y,\omega}[E]\big]\,,$$

which yields the last inequality of (1.26).

From (1.22) with $\overline{\omega}$, we analogously see that

$$I_U(\overline{\omega}) \leq \sum{}' e^{\ell\cdot(x+y)}\, P^U_{y,\overline{\omega}}[H_{\partial_- U_0} < H_{\partial_+ U_0}] + \sum{}'' e^{\ell\cdot(x+y)}$$
$$\leq c(\lambda)\Big[\sum{}' (P_{y,\overline{\omega}}[\widetilde{T}_{-L} < T_0]$$
$$+ P_{y,\overline{\omega}}[\sup_{i\geq 2, n \leq T_0 \wedge \widetilde{T}_{-L}} \big|(X_n - X_0)\cdot f_i| \geq M]\big) + \sum{}'' 1\Big]\,, \tag{1.27}$$

with $\sum''$ defined analogously as $\sum'$ in (1.26), imposing instead that $\sup_{i\geq 2} |y \cdot f_i| \geq \widetilde{L} - M$. From (1.26), (1.27), letting $\widetilde{L}$ and then M tend to infinity, and using translation invariance, we obtain:

$$P_0[\widetilde{T}_{-L} < D_+] \leq c(\lambda)\, P_{0,\overline{\omega}}[\widetilde{T}_{-(L-1)} < T_1]\,. \tag{1.28}$$

Note that $P_{0,\overline{\omega}}$ is the law of a random walk with drift $E_{0,\overline{\omega}}[X_1 - X_0] = \sum_1^d \sinh(\ell \cdot e_i) e_i / \sum_1^d \cosh(\ell \cdot e_i)$, that has a scalar product with $\widehat{\ell}$ bounded below by $c(\lambda)$, the right-hand-side of (1.28) is smaller than $c(\lambda)e^{-c(\lambda)L}$, see for instance (1.22) of [26]. The claim (1.16) easily follows.

We now turn to the proof of (1.17). We define for $L, \widetilde{L} > 1$,

$$\begin{aligned} &U_1 = \{z \in \mathbb{Z}^d,\ 0 < z \cdot f_1 < L,\ \sup_{i \ge 2} |z \cdot f_i| < \widetilde{L}\},\ U' = U_1 \cup \partial U_1\,, \\ &\partial_+ U_1 = \{z \in \partial U_1,\ z \cdot f_1 \ge L\},\ \partial_- U_1 = \{z \in \partial U_1,\ z \cdot f_1 \le 0\}\,. \end{aligned} \tag{1.29}$$

It follows from Dirichlet's principle that for $\omega \in \Omega$:

$$I_{U'}(\omega) = \sum_{\partial_- U_1} m_\omega^{U'}(x)\, P_{x,\omega}^{U'}[H_{\partial_+ U_1} < \widetilde{H}_{\partial_- U_1}],\ \text{ with} \tag{1.30}$$

$$I_{U'}(\omega) = \inf\{\mathcal{E}^{U'}(g,g)\ g_{|\partial_- U_1} = 1\ g_{|\partial_+ U_1} = 0\}\,, \tag{1.31}$$

and $P_{x,\omega}^{U'}$ stands for the canonical law of the reflected process in U' starting at x. We let $\pi_j(i)$, $1 \le j \le N(\omega)$, $1 \le i \le m_j$, denote a maximal collection of edge-disjoint open self-avoiding paths in U' starting in $\partial_+ U_1$, ending in $\partial_- U_1$. Then for g as in (1.31), summing over $1 \le j \le N(\omega)$, $1 \le i < m_j$; and using Cauchy-Schwarz's inequality, one finds:

$$\begin{aligned} N(\omega)^2 &= \Big(\sum_{i,j} g\big(\pi_j(i+1)\big) - g\big(\pi_j(i)\big)\Big)^2 \le \mathcal{E}^{U'}(g,g) \sum_{i,j} w_\omega^{-1}(\{\pi_j(i), \pi_j(i+1)\}) \\ &\le c(\lambda)\, \mathcal{E}^{U'}(g,g) \sum_{x \in U'} e^{-2\ell \cdot x} \le c(\lambda)\, \mathcal{E}^{U'}(g,g)\, |\partial_- U_1|\,. \end{aligned} \tag{1.32}$$

From (1.30), minimizing over g, and integrating over ω, we find:

$$c(\lambda)\, |\partial_- U_1|^{-1}\, \mathbb{E}[N(\omega)^2] \le \sum_{\partial_- U_1} \mathbb{E}\big[P_{x,\omega}^{U'}[H_{\partial_+ U_1} < \widetilde{H}_{\partial_- U_1}]\big]\,. \tag{1.33}$$

We pick $M > 1$, and observe that for $x \in \partial_- U_1$ with $\sup_{i \ge 2} |x \cdot f_1| < \widetilde{L} - M$, the expectation in the right-hand-side of (1.33) is smaller than:

$$\begin{aligned} &c(\lambda)(P_x[T_L < \widetilde{H}_{\partial_- U_1},\ \sup_{i \ge 2, n \le T_L} |(X_n - X_0) \cdot f_i| < M] \\ &\quad + P_0[\sup_{i \ge 2, n \le T_L \wedge \widetilde{T}_{-1}} |X_n \cdot f_i| \ge M - 1])\,. \end{aligned} \tag{1.34}$$

Letting $\widetilde{L}$ tend to infinity in (1.33), we obtain

$$c(\lambda) \varlimsup_{\widetilde{L} \to \infty} \frac{\mathbb{E}[N^2(\omega)]}{|\partial_- U_1|^2} \le P_0[T_L < D_-] + P_0\big[\sup_{i \ge 2, n \le T_L \wedge \widetilde{T}_{-1}} |X_n \cdot f_i| \ge M - 1\big]\,. \tag{1.35}$$

From Theorems 7.68 and 11.22 in Grimmett [14], we know that the left-hand-side of (1.35) is bigger than $c(\lambda)$. The claim (1.17) will follow once we show that the rightmost term of (1.35) tends to 0 as M tends to infinity. To this end note that for any $i \ge 2$, under P_0, once $|X_n \cdot f_i|$ reaches a new maximum, the walk has conditionally on its past a probability at least $c(\lambda, L)$ of exiting the strip $\{z : -1 < z \cdot f_1 < L\}$, within the next $c\,L$ steps, (with c in fact depending solely on d). The claim then easily follows. $\square$

We are now ready to prove directional transience.

Theorem 1.2.

$$\mathbb{P}\text{-}a.s., \textit{for all } x \in \mathcal{C},\ P_{x,\omega}[\lim_n X_n \cdot \ell = \infty] = 1\,. \tag{1.36}$$

Proof. Recall the notation (1.5), and observe it suffices to prove that

$$P_0[\lim_n X_n \cdot \ell = \infty \,|\, \mathcal{I}^-] = 1\,. \tag{1.37}$$

Indeed from the ergodic theorem, (1.37) and the uniqueness of the infinite cluster, we see that $\mathbb{P}$-a.s., for infinitely many $y \in \mathbb{Z}^d$, $P_{y,\omega}[\lim_n X_n \cdot \ell = \infty] = 1$, and $P_{y,\omega}[H_x < \infty] > 0$, for all $x \in \mathcal{C}$. The claim (1.36) then follows from the strong Markov property.

We thus prove (1.37). We pick $L > 1$, and note that

$$\begin{aligned} P_0[D_+ = \infty = \widetilde{T}_{-L}] &\le P_0[D_+ = \infty, 0 \text{ belongs to a finite cluster}] \\ &+ P_0[D_+ = \infty = \widetilde{T}_{-L},\ \sup_{i \ge 2, n \ge 0} |X_n \cdot f_i| = \infty] = 0\,, \end{aligned} \tag{1.38}$$

where we use for the last term a similar argument as for the control of the rightmost term of (1.35). Keeping in mind (1.16), we see that

$$P_0[D_+ < \infty] = 1\,. \tag{1.39}$$

We can then consider the sequence D_k, $k \ge 0$, of iterates of D_+:

$$D_0 = 0,\ D_{k+1} = D_+ \circ \theta_{D_k} + D_k,\ k \ge 0\,. \tag{1.40}$$

Note that P_0-a.s., the $(D_k)_{k \ge 0}$, are finite, increasing and $X_{D_k} \cdot \ell$ is non-decreasing. We first show that

$$P_0[\cdot\,|\,\mathcal{I}^-]\text{-a.s., } \{X_{D_k}, k \ge 0\} \text{ is an infinite subset of } \mathbb{Z}^d\,. \tag{1.41}$$

Indeed otherwise for some finite $A \subseteq \mathbb{Z}^d$, $P_0[\{X_{D_k}, k \ge 0\} = A \subseteq \mathcal{C}] > 0$. Since $\mathbb{P}$-a.s., on $\{A \subseteq \mathcal{C}\}$, for some (in fact any) $x_0 \in \mathcal{C}$, with $\sup_{x \in A} x \cdot \ell < x_0 \cdot \ell$, $\inf_{x \in A} P_{x,\omega}[H_{x_0} < \infty] > 0$, it would follow that $P_0[\sup_n X_n \cdot \ell > \sup_{k \ge 0} X_{D_k} \cdot \ell] > 0$, which is impossible.

We thus introduce V_i, $i \ge 0$, the sequence of successive times of visit of new sites by X_{D_k}, $k \ge 0$:

$$V_0 = 0,\ V_{i+1} = \inf\{D_k > V_i;\ X_{D_k} \ne X_{D_{k'}}, \text{for all } k' < k\}, \text{for } i \ge 0\,. \tag{1.42}$$

Let us then fix $e_0 \in \mathbb{Z}^d$, with $|e_0| = 1$, such that:

$$e_0 \cdot \widehat{\ell} = \max_{|e|=1, e \in \mathbb{Z}^d} e \cdot \widehat{\ell} \Big(\ge \frac{1}{\sqrt{d}} \Big)\,, \tag{1.43}$$

(for some directions ℓ, there may be more than one single choice for e_0). Then $P_0[\cdot\,|\,\mathcal{I}^-]$-a.s., for $i \ge 0$,

$$P_0[X_{V_i+1} = X_{V_i} + e_0 | \mathcal{F}_{V_i} \cap \mathcal{I}^-] = \sum_{x \in \mathbb{Z}^d} P_0[X_{V_i+1} = x + e_0,\ X_{V_i} = x | \mathcal{F}_{V_i} \cap \mathcal{I}^-]$$

so that from (0.1) and (1.43),

$$\ge \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} P_0[\omega(\{x, x+e_0\}) = 1,\ X_{V_i} = x | \mathcal{F}_{V_i} \cap \mathcal{I}^-], \tag{1.44}$$

and since under P_0, $\omega(\{x, x+e_0\})$ is independent of $\mathcal{F}_{V_i} \cap \{X_{V_i} = x\} \cap \mathcal{I}^-$,

$$= \sum_{x \in \mathbb{Z}^d} \frac{p}{2d} P_0[X_{V_i} = x | \mathcal{F}_{V_i} \cap \mathcal{I}^-] = \frac{p}{2d}\,.$$

Using Borel-Cantelli's lemma, cf. Durrett [12], p. 207, we find that

$$\begin{aligned} &P_0[\cdot\,|\,\mathcal{I}^-]\text{-a.s., } X_{V_i+1} = X_{V_i} + e_0, \text{ for infinitely many } i \geq 0,\\ &(\text{and thus } \lim_i X_{V_i} \cdot \ell = \infty)\,. \end{aligned} \tag{1.45}$$

We then choose $M' > M > 1$, and define, cf. (1.14) for the notations:

$$V' = T_{M'}, \text{ and for } k \geq 0,\ N_k = \sum_{n=0}^{k} 1\{X_n \cdot \widehat{\ell} \leq M\}\,. \tag{1.46}$$

With a slight variation of the argument in (1.44), we see that $P_0[\cdot\,|\,\mathcal{I}^-]$-a.s.,

$$\begin{aligned} &P_0\Big[\{X_{V'+1} = X_{V'} + e_0\} \cap \theta^{-1}_{V'+1}\Big(\bigcap_{L>1}\{T_L < D_-\}\Big)\,|\,\mathcal{F}_{V'} \cap \mathcal{I}^-\Big]\\ &\geq \sum_{x \in \mathbb{Z}^d} P_0\big[X_{V'} = x, \omega(\{x, x+e_0\})\\ &= 1\,|\,\mathcal{F}_{V'} \cap \mathcal{I}^-\big]\ \frac{1}{2d} P_0\big[\bigcap_{L>1}\{T_L < D_-\}\,|\,\omega(\{-e_0, 0\}) = 1\big]\\ &= \frac{p}{2d}\ P_0\big[\bigcap_{L>1}\{T_L < D_-\}\,|\,\omega(\{-e_0,0\}) = 1\big] \geq c(\lambda)\,, \end{aligned} \tag{1.47}$$

using (1.17) and an argument similar as below (1.26) in the last step. As a result, for $n \geq 0$,

$$P_0[N_\infty = \infty, N_{V'} \geq n\,|\,\mathcal{I}^-] \leq \big(1 - c(\lambda)\big)\ P_0[N_{V'} \geq n\,|\,\mathcal{I}^-]\,. \tag{1.48}$$

Letting M' and then n tend to infinity, it follows that

$$P_0[N_\infty = \infty\,|\,\mathcal{I}^-] = 0\,. \tag{1.49}$$

Since $M > 1$ is arbitrary, (1.37) and thus (1.36) follow. $\square$

2. The Renewal Structure

In this section we take advantage of the transience in the direction $\widehat{\ell}$ of the walk on the infinite cluster, see Theorem 1.2, and introduce certain regeneration times. These times enable us to construct a renewal structure for the walk under the measure P, see (0.9). The approach has a similar spirit to Shen [23], or Sznitman-Zerner [28], however there is a special twist due to the conditioning present in the definition of P. In particular the regeneration times defined here are configuration dependent.

We begin with some notations. For $x \in \mathbb{Z}^d$, $\omega \in \mathcal{I}_x$, cf. (1.6), $j \geq 0$, we consider

$$\begin{aligned} \mathcal{P}_{j,x}(\omega) = {} &\text{the set of open self-avoiding } \mathcal{C}\text{-valued paths, } (\pi(i))_{i\geq 0}, \text{ with}\\ &\pi(0) = x \text{ and } \pi(i) \in \mathcal{H}_x^-, \text{ for } i \geq j\,. \end{aligned} \tag{2.1}$$

Since $p > p_c$, see (0.7), percolation takes place on $\mathcal{H}_x^-$, see above (1.5), and $\mathbb{P}$-a.s., on $\mathcal{I}_x$, $\mathcal{P}_{j,x}(\omega)$ is not empty for large j. We thus define:

$$\begin{aligned} J_x(\omega) &= \inf\{j \geq 0;\ \mathcal{P}_{j,x}(\omega) \neq \emptyset\},\ \text{ if } \omega \in \mathcal{I}_x,\\ &= \infty, \text{ if } \omega \notin \mathcal{I}_x\,. \end{aligned} \tag{2.2}$$

We then introduce the sequence of configuration dependent stopping times and corresponding successive maxima of the walk in the direction $\widehat{\ell}$, cf. (1.14) for notations,

$$\begin{aligned}&W_0 = 0,\ m_0 = J_{X_0}(\omega) \le \infty, \text{ and by induction},\\ &W_{k+1} = 2 + T_{m_k} \le \infty,\ m_{k+1} = \sup\{X_n \cdot \widehat{\ell}, n \le W_{k+1}\} + 1 \le \infty, \quad \text{for all } k \ge 0.\end{aligned} \tag{2.3}$$

As a result of Theorem 1.2, we see that

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s., } W_k < \infty, \text{ for all } k \ge 0. \tag{2.4}$$

With the choice of e_0 as in (1.43) we introduce the collection of bonds:

$$\begin{aligned}B = \{b \in B_d;\, b = \{&-e_0, e - e_0\}, \text{ with } e \text{ any unit vector}\\ &\text{of } \mathbb{Z}^d \text{ such that } e \cdot \widehat{\ell} = e_0 \cdot \widehat{\ell}\},\end{aligned} \tag{2.5}$$

as well as the configuration dependent stopping times

$$\begin{aligned}S_1 = \inf\{&W_k;\ k \ge 1,\ X_{W_k} = X_{W_k - 1} + e_0 = X_{W_k - 2} + 2e_0, \text{ and}\\ &\omega(b) = 1, \text{ for all } b \in B + X_{W_k}\}.\end{aligned} \tag{2.6}$$

Lemma 2.1.

$$\mathbb{P}\text{-}a.s., \text{ for all } x \in \mathcal{C},\ P_{x,\omega}\text{-}a.s.,\ S_1 < \infty. \tag{2.7}$$

Proof. Observe that for $k \ge 0$,

$$\begin{aligned}P_0[\mathcal{I}, S_1 > W_{k+1}] = \sum_{x \in \mathbb{Z}^d} &\mathbb{E}[\mathcal{I}, P_{0,\omega}[S_1 > W_k, X_{T_{m_k}} = x](1 -\\ &P_{x,\omega}[X_1 = x + e_0, X_2 = x + 2e_0]\, 1_{\mathcal{O}_{B+x+2e_0}})].\end{aligned} \tag{2.8}$$

Moreover $\mathbb{P}$-a.s., $1_{\mathcal{I}}\, P_{0,\omega}[S_1 > W_k, X_{T_{m_k}} = x] = 1_{\{J_0 \le x \cdot \widehat{\ell}\}} P_{0,\omega}[S_1 > W_k, X_{T_{m_k}} = x]$ is $\sigma(\omega(b), b \ne \{x, x + e_0\}, b \notin B + x + 2e_0)$-measurable. On the other hand the second factor in the above expectation is smaller than $1 - (\frac{1}{2d})^2\, 1_{\mathcal{O}_{B+x+2e_0} \cup \{\{x, x+e_0\}\}}$, which is independent from the above σ-algebra. Therefore:

$$\begin{aligned}P_0[\mathcal{I}, S_1 > W_{k+1}] &\le P_0[\mathcal{I}, S_1 > W_k]\Big(1 - \Big(\frac{1}{2d}\Big)^2 p^{|B|+1}\Big)\\ &\le \Big(1 - \Big(\frac{1}{2d}\Big)^2 p^{|B|+1}\Big)^{k+1},\end{aligned} \tag{2.9}$$

for $k \ge 0$, using induction in the last step. As a result we see that P_0-a.s., in $\mathcal{I}$, $S_1 < \infty$, and (2.7) follows. □

We then define

$$D = \inf\{n \ge 0,\ X_n \cdot \widehat{\ell} < X_0 \cdot \widehat{\ell}\}, \tag{2.10}$$

as well as the configuration dependent stopping times S_k, $k \geq 0$, R_k, $k \geq 1$, and the levels M_k, $k \geq 0$:

$$\begin{aligned} &S_0 = 0,\ M_0 = X_0 \cdot \widehat{\ell}, \text{ and for } k \geq 0\,, \\ &S_{k+1} = S_1 \circ \theta_{T_{M_k}} + T_{M_k} \leq \infty,\ R_{k+1} = D \circ \theta_{S_{k+1}} + S_{k+1} \leq \infty\,, \\ &M_{k+1} = \sup_{n \leq R_{k+1}} X_n \cdot \widehat{\ell} + 1 \leq \infty\,. \end{aligned} \tag{2.11}$$

In view of Theorem 1.2, and Lemma 2.1, it follows that

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s., for } k \geq 1,\ S_{k+1} < \infty \text{ on } \{R_k < \infty\}\,. \tag{2.12}$$

We can now define the basic regeneration time:

$$\tau_1 = S_K, \text{ with } K = \inf\{k \geq 1,\ S_k < \infty \text{ and } R_k = \infty\}\,. \tag{2.13}$$

Observe that in contrast to [28, 23], τ_1 is configuration dependent.

Lemma 2.2.

$$\mathbb{P}\text{-}a.s., \textit{for all } x \in \mathcal{C},\ P_{x,\omega}\text{-}a.s.,\ \tau_1 < \infty\,. \tag{2.14}$$

Proof. It is straightforward to deduce from (1.17), that

$$P_0[D = \infty | \mathcal{O}_B] \geq P_0[D_- = \infty \,|\, \mathcal{O}_B] \geq c(\lambda)\,. \tag{2.15}$$

We now use an argument with a similar flavor as in (2.8). For $k \geq 1$, we have:

$$P_0[\mathcal{I},\, R_k < \infty] = \sum_{x \in \mathbb{Z}^d} \mathbb{E}\big[\mathcal{I},\, P_{0,\omega}[S_k < \infty,\, X_{S_k} = x]\, P_{x,\omega}[D < \infty]\big]\,. \tag{2.16}$$

We introduce

$$E_x^+ = \text{the collection of } b = \{y, z\} \text{ in } B_d \text{ with } y \text{ or } z \text{ in } \mathcal{H}_x^+ \text{ and } y, z \neq x - e_0, \tag{2.17}$$

$$E_x^- = B_d \backslash E_x^+\,, \tag{2.18}$$

and observe that up to a $\mathbb{P}$-negligible set, $1_{\mathcal{I}}\ P_{0,\omega}[S_k < \infty, X_{S_k} = x]$ is $\sigma(\omega(b),$ $b \in E_x^-)$-measurable. Thus conditioning on this σ-algebra in the expectation in (2.16), we obtain

$$\begin{aligned} P_0[\mathcal{I},\, R_k < \infty] &= P_0[\mathcal{I},\, S_k < \infty]\, P_0[D < \infty \,|\, \mathcal{O}_B] \\ &\leq (1 - c(\lambda))^k,\ \text{ with } c(\lambda) \in (0, 1)\,, \end{aligned} \tag{2.19}$$

using induction and (2.15). Thus P_0-a.s., on $\mathcal{I}$, $\tau_1 < \infty$, and (2.14) easily follows. □

Under P_0, $\mathcal{I}_{-e_0}^-$ and $\mathcal{O}_B \cap \{D = \infty\}$ are independent and have positive probability, see (1.5), (2.15). The following conditional measure, that plays an important role for the renewal property, is thus well defined:

$$Q = P_0[\cdot \,|\, \mathcal{I}_{-e_0}^-, \mathcal{O}_B, D = \infty]\,. \tag{2.20}$$

We denote by E^P, E^Q the expectations under the respective measures P and Q. The next proposition is the key step for the renewal property.

Proposition 2.3. *Let f be a bounded $\sigma(\omega(b), b \in E_0^+)$-measurable function, $\widetilde{f}$ be a bounded $\sigma(\omega(b), b = \{x, y\}, x \cdot \widehat{\ell} \leq 0, y \cdot \widehat{\ell} \leq 0)$-measurable function and g, h be bounded measurable functions on $(\mathbb{Z}^d)^{\mathbb{N}}$, then*

$$E^P[\widetilde{f}\, h(X_{\tau_1 \wedge \cdot})\, g(X_{\tau_1 + \cdot} - X_{\tau_1})\, f \circ t_{X_{\tau_1}}] = E^P[\widetilde{f}\, h(X_{\tau_1 \wedge \cdot})]\, E^Q[g(X_{\cdot}) f], \quad (2.21)$$

(*cf.* (0.9)*, and above* (1.6) *for the notations*).

Proof. The left-hand-side of (2.21) multiplied by $P_0(\mathcal{I})$ equals:

$$\sum_{k \geq 1} E_0[\mathcal{I}, \tau_1 = S_k, \widetilde{f}\, h(X_{S_k \wedge \cdot})\, g(X_{S_k + \cdot} - X_{S_k})\, f \circ t_{X_{S_k}}] = \sum_{k \geq 1, x \in \mathbb{Z}^d} \mathbb{E}\big[\mathcal{I}, \widetilde{f} \\ \times E_{0,\omega}[h(X_{S_k \wedge \cdot}), S_k < \infty, X_{S_k} = x]\, f \circ t_x\, E_{x,\omega}[D = \infty, g(X_{\cdot} - x)]\big]. \quad (2.22)$$

The above expression will not change if we replace ω in the $E_{x,\omega}$-expectation with ω_x^+, which coincides with ω on E_x^+ and satisfies $\omega^+(b) = 1$, for $b \in E_x^-$. Further note that $1_{\mathcal{I}}\, \widetilde{f}\, E_{0,\omega}[h(X_{S_k \wedge \cdot}), S_k < \infty, X_{S_k} = x]$ is $\sigma(\omega(b), b \in E_x^-)$-measurable. Thus the above equals

$$\sum_{k \geq 1, x \in \mathbb{Z}^d} \mathbb{E}\big[\mathcal{I}, \widetilde{f}\, E_{0,\omega}[h(X_{S_k \wedge \cdot}), X_{S_k} = x]\big]\, P_0[D = \infty | \mathcal{O}_B] E_0[f\, g(X_{\cdot}) | \mathcal{O}_B, D = \infty].$$

Using the above equality with $f = 1$, $g = 1$, we see that the above equals

$$P_0(\mathcal{I})\, E^P[\widetilde{f}\, h(X_{\tau_1 \wedge \cdot})]\, E_0[f\, g(X_{\cdot}) \,|\, \mathcal{O}_B, D = \infty].$$

Our claim (2.21) follows straightforwardly since the last term coincides with E^Q $[f\, g(X)]$. □

With the help of the above proposition we can define the sequence of P-a.s. finite times $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_k < \dots$, via the following procedure (with hopefully obvious notations):

$$\tau_{k+1} = \tau_1 + \tau_k(X_{\tau_1 + \cdot} - X_{\tau_1}, t_{X_{\tau_1}} \omega),\ k \geq 0. \quad (2.23)$$

Note that for any x, P-a.s., $\{X_{\tau_1} = x\} \subseteq \mathcal{I}_x^- \subseteq \{J_x = 0\}$, and from (2.3), (2.6), (2.11), $\tau_k(X_{\tau_1 + \cdot} - X_{\tau_1}, t_{X_{\tau_1}} \omega)$, P-a.s. coincides with a measurable function of $X_{\tau_1 + \cdot} - X_{\tau_1}$ and $(t_{X_{\tau_1}} \omega)(b)$, $b \in E_0^+$, thus in particular the P-a.s. finiteness of the τ_k follows from Proposition 2.3. We now come to the renewal property:

Theorem 2.4. *Under P, $(X_{\tau_1 \wedge \cdot})$, $(X_{(\tau_1 + \cdot) \wedge \tau_2} - X_{\tau_1}), \dots, (X_{(\tau_k + \cdot) \wedge \tau_{k+1}} - X_{\tau_k}), \dots,$ are independent and except for the first process distributed like $(X_{\tau_1 \wedge \cdot})$ under Q.*

Proof. Consider $k \geq 2$, and $h_1, \dots, h_k$ bounded measurable functions on $(\mathbb{Z}^d)^{\mathbb{N}}$. We recall that $\tau_0 = 0$, see above (2.23). It follows from Proposition 2.3 that

$$E^P\Big[\prod_{i=1}^{k} h_i(X_{(\tau_{i-1} + \cdot) \wedge \tau_i} - X_{\tau_{i-1}})\Big] \\ = E^P[h_1(X_{\tau_1 \wedge \cdot})] E^Q\Big[\prod_{i=2}^{k} h_i(X_{(\tau_{i-2} + \cdot) \wedge \tau_{i-1}} - X_{\tau_{i-2}})\Big]. \quad (2.24)$$

Observe that P-a.s., $\{D = \infty\} = \{D > \tau_1\}$ and $Q = P[\cdot \mid D > \tau_1, \mathcal{O}_B, \mathcal{I}^-_{-e_0}]$, so the last factor in the right-hand-side of (2.24) equals

$$E^P[D > \tau_1, \mathcal{O}_B, \mathcal{I}^-_{-e_0},\ h_2(X_{\cdot \wedge \tau_1}) \\ \times \prod_{2 \le i < k} h_{i+1}(X_{(\tau_{i-1}+\cdot)\wedge \tau_i} - X_{\tau_{i-1}})]/P[D > \tau_1, \mathcal{O}_B, \mathcal{I}^-_{-e_0}]\,.$$

Applying Proposition 2.3 again, we see that the above expression equals

$$E^Q[h_2(X_{\tau_1 \wedge \cdot})]\, E^Q[\prod_{3 \le i \le k} h_i(X_{(\tau_{i-3}+\cdot)\wedge \tau_{i-2}} - X_{\tau_{i-3}})]\,,$$

(where the last factor is of course absent when $k < 3$). Thus a repeated application of Proposition 2.3 yields that the left-hand-side of (2.24) equals:

$$E^P[h_1(X_{\tau_1 \wedge \cdot})] \prod_{i=2}^{k} E^Q[h_i(X_{\tau_1 \wedge \cdot})]\,.$$

Our claim then follows. □

We close this section with a stochastic domination result that will be useful in the next section, when controlling moments of $\sup_{n \le \tau_1} |X_n|$ under Q. We introduce the probability

$$\widetilde{Q} = P_0[\cdot \,|\, \mathcal{I}^-_{-e_0},\ \mathcal{O}_B, D < \infty]\,. \tag{2.25}$$

Proposition 2.5. *Under $P_0[\cdot \,|\, \mathcal{I}^-_{-e_0}, \mathcal{O}_B]$, $X_{\tau_1} \cdot \widehat{\ell}$ is stochastically dominated by Σ_J, where $\Sigma_0 = 0$, and for $k \ge 1$:*

$$\Sigma_k = (2 + \overline{M}_1 + 4\overline{H}_1) + \cdots + (2 + \overline{M}_k + 4\overline{H}_k)\,, \tag{2.26}$$

for independent variables $\overline{M}_i, \overline{H}_i, i \ge 1, J$, with $\overline{M}_i, \overline{H}_i$ respectively distributed like

$$M = \sup_{n \le D} (X_n \cdot \widehat{\ell} - X_0 \cdot \widehat{\ell}) \text{ under } \widetilde{Q}, \text{ and} \tag{2.27}$$

$$H \text{ a geometric variable on } \mathbb{N}^* \ (\stackrel{\text{def}}{=} \mathbb{N}\backslash\{0\}) \text{ with parameter } 1 - p^{|B|+1}/(2d)^2\,, \tag{2.28}$$

and J an independent geometric variable on $\mathbb{N}^*$ with parameter $P_0[D < \infty \,|\, \mathcal{O}_B]$.

Proof. We consider a non-decreasing, non-negative, bounded function f on $\mathbb{R}$ and for simplicity write

$$\mathcal{A} = \mathcal{I}^-_{-e_0} \cap \mathcal{O}_B\,. \tag{2.29}$$

With a similar calculation as in (2.22), we find:

$$E_0[\mathcal{A}, f(X_{\tau_1} \cdot \widehat{\ell})] = P_0[D = \infty \,|\, \mathcal{O}_B] \sum_{k \ge 1} E_0[\mathcal{A}, S_k < \infty, f(X_{S_k} \cdot \widehat{\ell})]\,. \tag{2.30}$$

We thus consider for $k \geq 2$:

$$
\begin{aligned}
&E_0[\mathcal{A}, S_k < \infty, f(X_{S_k} \cdot \widehat{\ell})] \\
&\overset{(2.11)}{\leq} E_0\big[\mathcal{A}, R_{k-1} < \infty, f\big(M_{k-1} + 1 + \big((X_{S_1} - X_0) \cdot \widehat{\ell}\big) \circ \theta_{T_{M_{k-1}}}\big)\big] \\
&\leq \sum_{j \geq 1} E_0[\mathcal{A}, R_{k-1} < \infty,\ f(M_{k-1} + 4j + 1),\ \theta^{-1}_{T_{M_{k-1}}}(S_1 = W_j)] \\
&= \sum_{j \geq 1} E_0[\mathcal{A}, R_{k-1} < \infty,\ f(M_{k-1} + 4(j+1) + 1) \\
&\quad - f(M_{k-1} + 4j + 1),\ \theta^{-1}_{T_{M_{k-1}}}(S_1 > W_j)] + E_0[\mathcal{A}, R_{k-1} < \infty,\ f(M_{k-1} + 5)],
\end{aligned}
$$

so using similar considerations as in (2.8), (2.9)

$$
\begin{aligned}
&\leq \sum_{j \geq 1} E_0[\mathcal{A}, R_{k-1} < \infty,\ f(M_{k-1} + 4(j+1) + 1) \\
&\quad - f(M_{k-1} + 4j + 1)]\Big(1 - \frac{p^{|B|+1}}{(2d)^2}\Big)^j + E_0[\mathcal{A}, R_{k-1} < \infty, f(M_{k-1} + 5)] \\
&= \overline{E} \times E_0[\mathcal{A}, R_{k-1} < \infty,\ f(M_{k-1} + 4\overline{H}_k + 1)]\,, \qquad (2.31)
\end{aligned}
$$

where $\overline{P}$ stands for the probability governing the variables $\overline{M}_i$, $\overline{H}_i$, $i \geq 1$, J, and $\overline{E}$ for the corresponding expectation. In view of (2.11), the above expression is smaller than:

$$
\begin{aligned}
&\overline{E} \times E_0[\mathcal{A}, S_{k-1} < \infty, D \circ \theta_{S_{k-1}} < \infty,\ f(X_{S_{k-1}} \cdot \widehat{\ell} + M \circ \theta_{S_{k-1}} + 4\overline{H}_k + 2)] \\
&\quad = \sum_{x \in \mathbb{Z}^d} \overline{E} \times \mathbb{E}\big[\mathcal{A}, P_{0,\omega}[S_{k-1} < \infty,\ X_{S_{k-1}} = x] E_{x,\omega}[f(x \cdot \widehat{\ell} + M \\
&\qquad + 4\overline{H}_k + 2), D < \infty]\big],
\end{aligned}
$$

using a similar argument as in (2.22)

$$= \overline{E} \times E_0[\mathcal{A}, S_{k-1} < \infty, f(X_{S_{k-1}} \cdot \widehat{\ell} + \overline{M}_k + 4\overline{H}_k + 2)]\, P_0[D < \infty \,|\, \mathcal{O}_B]$$

and by induction

$$\leq \overline{E} \times E_0[\mathcal{A}, f(X_{S_1} \cdot \widehat{\ell} + \Sigma_k - \Sigma_1)]\, P_0[D < \infty \,|\, \mathcal{O}_B]^{k-1}\,. \qquad (2.32)$$

Repeating the argument in (2.31), we see that the left-hand-side of (2.31), for $k \geq 2$, is smaller than:

$$P_0(\mathcal{A})\, \overline{E}[f(\Sigma_k)]\, P_0[D < \infty \,|\, \mathcal{O}_B]^{k-1}\,,$$

and this inequality remains true for $k = 1$. Inserting these inequalities in (2.30), we readily obtain our claim. $\square$

3. Weak Anisotropy and Ballistic Behavior

Our main object in this section is to prove for small λ, a law of large numbers with non-degenerate limiting velocity and a functional central limit theorem governing the corrections to the law of large numbers, see Theorem 3.4. With the help of the renewal property in Theorem 2.4, the main task is to show that for small λ, τ_1 has a finite second moment under the measure Q of (2.20). We will obtain such an estimate by controlling the $\mathbb{P}$-probability of occurrence of certain low principal Dirichlet eigenvalues in suitable large boxes, cf. Lemma 4.2, and the exit measure of the walk from certain large boxes under P, cf. Lemma 4.1.

We will be interested in large boxes of the form, (see (1.18) for the notation),

$$U = \{x \in \mathbb{Z}^d,\ |x \cdot f_1| < L,\ \sup_{2 \le i \le d} |x \cdot f_i| < \widetilde{L}\},\ L,\ \widetilde{L} > 1\,, \tag{3.1}$$

specifically we will later choose L large and $\widetilde{L} \approx L^{\frac{7}{2}}$, cf. (3.28) below. We denote by $\partial_+ U$ the part of the boundary of U:

$$\partial_+ U = \{x \in \partial U,\ x \cdot f_1 \ge L\}\,. \tag{3.2}$$

An important role will be played by the principal Dirichlet eigenvalue of $I - R$, in $U \cap \mathcal{C}$, for $\omega \in \Omega_1$, cf. (1.3), (1.10):

$$\begin{aligned} \Lambda_\omega(U) &= \inf\{\mathcal{E}(f,f),\ f_{|(U\cap\mathcal{C})^c} = 0,\ \|f\|_{L^2(m)} = 1\},\ \text{when } U \cap \mathcal{C} \neq \emptyset\,,\\ &= \infty,\ \text{by convention when } U \cap \mathcal{C} = \emptyset\,. \end{aligned} \tag{3.3}$$

Note that when $U \cap \mathcal{C} \neq \emptyset$, $\Lambda_\omega(U) \le 1$. We will derive for suitably large boxes U, upper bounds on $\mathbb{P}[\Lambda_\omega(U) \le \gamma]$ for small γ (of order L^{-5}, see (3.28)) and on $P[X_{T_U} \notin \partial_+ U]$. These bounds will be instrumental for the control of moments of τ_1 under Q.

We are first going to derive a lower bound for $\Lambda_\omega(U)$ in terms of a geometric quantity, and provide an upper bound on $P_0[X_{T_U} \notin \partial_+ U]$ in terms of $\mathbb{P}[\Lambda_\omega(U) \le \gamma]$, for a suitable γ. The geometric quantity mentioned above, stems from renormalization techniques in percolation, see for instance chapter 7 Sect. 4 in [14], as we now explain.

If for all $1 \le i,\ j \le d$, $|\widehat{\ell} \cdot e_i| \le 2|\widehat{\ell} \cdot e_j|$, we choose u, u' orthogonal of the form $\pm e_i$, $1 \le i \le d$, with $u \cdot \widehat{\ell} > 0$, $u' \cdot \widehat{\ell} > 0$, and define the two-dimensional discrete quadrant:

$$\widehat{\mathbb{L}} = \mathbb{N}u + \mathbb{N}u'\,. \tag{3.4}$$

On the other hand, if for some j, in the notations of (1.43), $\widehat{\ell} \cdot e_0 > 2|\ell \cdot e_j|$, we choose $u = e_0$, $u' = e_j$, and define instead

$$\widehat{\mathbb{L}} = \{au + bu',\ 0 \le |b| \le a\}\,. \tag{3.5}$$

It then follows for instance from static renormalization, see Theorems 7.61, 7.65 in [14], that we can choose an integer K and a number c depending only on d, p, so that setting

$$\mathbb{L} = K\,\widehat{\mathbb{L}} + [-K, K]^d\,, \tag{3.6}$$

$$\mathbb{P}\text{-a.s., there is an infinite cluster in } y + \mathbb{L}, \text{ for each } y \in \mathbb{Z}^d\,, \tag{3.7}$$

(i.e. for each y the restriction of ω to bonds between sites in $y + \mathbb{L}$, induces an infinite connected component), and

$$\mathbb{E}[e^{c\Delta}] < 2,\ \text{with} \tag{3.8}$$

$$\Delta(\omega) = \inf\{k \ge 0,\ [-k, +k]^d \text{ meets an infinite cluster of } \mathbb{L}\}\,. \tag{3.9}$$

Thus for $\omega \in \Omega_1$, see (1.3), and $y \in \mathcal{C}$, we can define

$$D(y,\omega) = \text{the minimal distance in } \mathcal{C} \text{ of } y \text{ to an infinite cluster in } y + \mathbb{L}\,, \tag{3.10}$$

(i.e. the minimal number of steps of a nearest neighbor self-avoiding path in $\mathcal{C}$ along open edges, starting at y and ending in an infinite cluster of $y + \mathbb{L}$), and for $L, \widetilde{L} > 1$:

$$D(U,\omega) = \max_{y \in \mathcal{C}\cap U} D(y,\omega) \; (= 0, \text{ by convention when } \mathcal{C} \cap U = \emptyset)\,. \tag{3.11}$$

We are now ready to prove

Lemma 3.1. $(0 < \lambda \le 1)$

$$\Lambda_\omega(U) \ge c\, \frac{e^{-2\lambda D(U,\omega)}}{D(U,\omega)^{2d} + L^4}\,, \; \textit{for } L, \widetilde{L} > 1,\, \omega \in \Omega_1\,. \tag{3.12}$$

For $L > 1$, $0 < \gamma \le 1$, $\widetilde{L} = 4L/\sqrt{\gamma}$, *with the notation* (0.9),

$$P[X_{T_U} \notin \partial_+ U] \le \mathbb{P}[\Lambda_\omega(U) \le \gamma] + c\, L^d\, \gamma^{-\frac{(d+1)}{2}}\, e^{-\lambda L} \tag{3.13}$$

(*we refer to the end of the introduction for the convention about constants*).

Proof. We begin with the proof of (3.12). Without loss of generality, we assume that $\mathcal{C} \cap U \neq \emptyset$. For each $x \in \mathcal{C} \cap U$, we pick a self-avoiding open path $(\pi_x(i))_{0 \le i \le \ell_x}$, starting at x and remaining in U except for the terminal point which lies in ∂U. We denote by ℓ_x the length of this path. We then define the maximal backtracking of these paths in the direction $\widehat{\ell}$:

$$H = \max_{x \in \mathcal{C}\cap U} \max_i (x - \pi_x(i)) \cdot \widehat{\ell}\,. \tag{3.14}$$

Following a now classical argument, see Saloff-Coste [22], p. 369, we write for f as in (3.3),

$$1 = \sum_x f^2(x)\, m_\omega(x) = \sum_x \Big\{ \sum_i f\big(\pi_x(i+1)\big) - f\big(\pi_x(i)\big)\Big\}^2 m_\omega(x)$$

$$\overset{\text{Cauchy−Schwarz}}{\le} \sum_x \ell_x \Big\{ \sum_i \big[f(\pi_x(i+1)\big) - f\big(\pi_x(i)\big)\big]^2 \Big\}\, m_\omega(x)\,. \tag{3.15}$$

Note that using $\lambda \le 1$, we find that $m_\omega(x)/w_\omega(\{\pi_x(i),\, \pi_x(i+1)\}) \le c\, e^{2\lambda H}$, see (0.3), (1.8), and hence the above expression is smaller than

$$c\, e^{2\lambda H} \sum_{b=\{y,z\}} \big(f(z) - f(y)\big)^2\, w(b,\omega) \times \max_b \sum_{x \in \mathcal{C}\cap U,\, b \in \pi_x} \ell_x\,,$$

with the notation $b \in \pi_x$, meaning that $b = \{\pi_x(i), \pi_x(i+1)\}$, for some i. As a result we obtain the lower bound:

$$\Lambda_\omega(U) \ge c\, \frac{e^{-2\lambda H}}{\max_b \sum_{x \in \mathcal{C}\cap U,\, b\in\pi_x} \ell_x}\,. \tag{3.16}$$

We now specify the choice of paths π_x. For each $x \in \mathcal{C} \cap U$ we pick an open path in $\mathcal{C}$ of length $D(x,\omega)$ connecting x to an infinite cluster of $x + \mathbb{L}$, see (3.10), and then continue this path with an infinite self-avoiding open path in $x + \mathbb{L}$. At some point the concatenation of these two paths exits U, and we can extract a self-avoiding path between x and this exit point, and thus obtain the desired π_x. Observe that

i) $H \le D(U,\omega) + c$,

ii) $\ell_x \le D(U,\omega) + c\,L^2$, for $x \in \mathcal{C} \cap U$, (recall $\widehat{\mathbb{L}}$ is in essence two-dimensional),

iii) for $b = \{y,z\}$, $y,z \in U$: $\#\{x \in \mathcal{C} \cap U, b \in \pi_x\}$

$$\begin{aligned} &\le \#\{x \in \mathcal{C} \cap U,\ y \text{ or } z \in x + \mathbb{L}\} + \#\{x \in \mathcal{C} \cap U,\ \sum_1^d |x_i - y_i| \le D(U,\omega)\} \\ &\le c\big(L^2 + D(U,\omega)^d\big). \end{aligned} \tag{3.17}$$

Note that

$$\sum_{x \in \mathcal{C} \cap U, b \in \pi_x} \ell_x \le c(D(U,\omega) + L^2)(D(U,\omega)^d + L^2) \le c(D(U,\omega)^{2d} + L^4),$$

and $\lambda \le 1$, so that (3.12) follows from (3.16), (3.17). We then turn to the proof of (3.13). For $n \ge 1$, we have

$$\begin{aligned} P[X_{T_U} \notin \partial_+ U] \le\ & \mathbb{P}[\Lambda_\omega(U) \le \gamma] + P[T_U > n, \Lambda_\omega(U) > \gamma] \\ & + P[X_{T_U} \notin \partial_+ U,\ T_U \le n]. \end{aligned} \tag{3.18}$$

For $\omega \in \mathcal{I}$, see (0.8), one sees with the help of Perron-Frobenius' theorem that $1_{U \cap \mathcal{C}}\, R\, 1_{U \cap \mathcal{C}}$ has an operator norm on $L^2(m)$ given by its maximum positive eigenvalue, namely $1 - \Lambda_\omega(U)$. So for $n \ge 1$, and $x \in \mathcal{C} \cap U$, using the spectral theorem we find:

$$\begin{aligned} & m_\omega(x)\, P_{x,\omega}[T_U > n] \\ & \quad = (1_{\{x\}}, (1_{U\cap\mathcal{C}}\, R\, 1_{U\cap\mathcal{C}})^n\, 1_U)_{L^2(m)} \le \sqrt{m_\omega(x)}\, \sqrt{m_\omega(U)}\, \big(1 - \Lambda_\omega(U)\big)^n, \end{aligned}$$

and hence

$$P_{x,\omega}[T_U > n] \le \sqrt{\frac{m_\omega(U)}{m_\omega(x)}}\; e^{-n\Lambda_\omega(U)},\ \omega \in \mathcal{I},\ x \in \mathcal{C} \cap U,\ n \ge 1. \tag{3.19}$$

Thus the second term in the right-hand-side of (3.18), using $\lambda \le 1$, is smaller than

$$c\, L^{\frac{1}{2}}\, \widetilde{L}^{\frac{(d-1)}{2}}\, e^{\lambda L - n\gamma}. \tag{3.20}$$

Further for $\omega \in \Omega_1$, $x, y \in \mathcal{C}$, with $d_{\mathcal{C}}(x,y)$ the distance on the infinite cluster between x and y, i.e. the minimum number of steps of an open path between x and y, Carne's estimate [8] yields:

$$P_{x,\omega}[X_n = y] \le 2\sqrt{\frac{m_\omega(y)}{m_\omega(x)}}\ \exp\Big\{-\frac{d_{\mathcal{C}}(x,y)^2}{2n}\Big\},\ n \ge 1, \tag{3.21}$$

and hence the last term in the right-hand-side of (3.18) is smaller than:

$$\sum_{k\le n} c\Big(\widetilde{L}^{(d-1)}\, e^{-\lambda L} + L\widetilde{L}^{(d-2)} \exp\Big\{-\frac{\widetilde{L}^2}{2k}+\lambda L\Big\}\Big)$$

$$\le c\, n\Big(\widetilde{L}^{(d-1)}\, e^{-\lambda L} + L\widetilde{L}^{(d-2)} \exp\Big\{-\frac{\widetilde{L}^2}{2n}+\lambda L\Big\}\Big). \tag{3.22}$$

If we now choose $n=[\frac{2L}{\gamma}]+1$, $\widetilde{L}=\frac{4L}{\sqrt{\gamma}}$, (note that $\frac{2L}{\gamma}\ge 1$, so $n\le\frac{4L}{\gamma}$), coming back to (3.18), (3.20), (3.22), we obtain (3.13). □

We now derive an upper bound on the $\mathbb{P}$-probability of occurrence of certain low eigenvalues, see (3.13).

Lemma 3.2. $(0<\lambda\le 1)$. *For $L>1$, $0<\gamma\le 1$, $\widetilde{L}=4L/\sqrt{\gamma}$,*

$$\mathbb{P}[\Lambda_\omega(U)\le\gamma]\le c\, L^d\,\gamma^{-\frac{(d-1)}{2}}\,[(c\,\gamma\, L^4)^{\frac{c}{\lambda}}+e^{-cL^{2/d}}]\,. \tag{3.23}$$

Proof. From the results in Antal-Pisztora [2], see Lemma 2.14 in [10] for the precise version we use here, there is a suitable $\rho(d,p)>0$, such that

$$\mathbb{P}[\text{ for some } z,z'\in[-k,k]^d\cap\mathcal{C},\, d_{\mathcal{C}}(z,z')>\rho k]\le e^{-ck},\ \text{for } k\ge 0, \tag{3.24}$$

with the same notation as in (3.21). Hence using (3.8), we see that for $u>0$,

$$\mathbb{P}[\mathcal{I},D(0,\omega)>u]\le\mathbb{P}\Big[\mathcal{I},D(0,\omega)>u,\ \Delta(\omega)\le\frac{u}{\rho}\Big]+\mathbb{P}\Big[\Delta(\omega)>\frac{u}{\rho}\Big]\le c\,e^{-cu}\,. \tag{3.25}$$

Thus for U as above we find:

$$\begin{aligned}
\mathbb{P}[\Lambda_\omega(U)\le\gamma] &\overset{(3.12)}{\le} \mathbb{P}\Big[c\,\frac{e^{-2\lambda\, D(U,\omega)}}{D(U,\omega)^{2d}+L^4}\le\gamma,\ U\cap\mathcal{C}\ne\emptyset\Big]\\
&\le\mathbb{P}[D(U,\omega)^{2d}\ge L^4,U\cap\mathcal{C}\ne\emptyset]+\mathbb{P}[e^{-2\lambda\, D(U,\omega)}\le c\,\gamma\, L^4,U\cap\mathcal{C}\ne\emptyset]\\
&\le|U|\Big(\mathbb{P}[D(0,\omega)\ge L^{2/d},\mathcal{I}]+\mathbb{P}\Big[D(0,\omega)\ge\frac{1}{2\lambda}\ \log\Big(\frac{1}{c\gamma L^4}\Big),\mathcal{I}\Big]\Big)\\
&\overset{(3.25)}{\le} c\,L^d\,\gamma^{-\frac{(d-1)}{2}}\,(e^{-c\,L^{2/d}}+(c\,\gamma\,L^4)^{c/\lambda})\,,
\end{aligned} \tag{3.26}$$

which proves (3.23). □

We are now ready to derive moment estimates on τ_1. The definition of Q appears in (2.20).

Proposition 3.3. *There exist a non-increasing sequence of constants λ_m, $m\ge 1$, in $(0,1]$ depending on d and p such that for $0<\lambda\le\lambda_m$:*

$$E^Q[\tau_1^m]<\infty\,. \tag{3.27}$$

Proof. We begin with a similar estimate for $\sup_{n \le \tau_1} |X_n|$ in place of τ_1. We write for $L > 1$,

$$U_L = U \text{ in (3.1), with } \widetilde{L} = 4L^{\frac{7}{2}} = 4L/\sqrt{\gamma},\ \gamma = L^{-5}\,. \tag{3.28}$$

Then for $u > 0$, with the notation (1.18), (2.29),

$$P_0\Big[\sup_{n \le \tau_1, 1 \le i \le d} |X_n \cdot f_i| \ge u, \mathcal{A}\Big] \le P_0\Big[\sup_{n \le \tau_1, 1 \le i \le d} |X_n \cdot f_i| \ge u,\ X_{\tau_1} \cdot \widehat{\ell} < \Big(\frac{u}{4}\Big)^{\frac{2}{7}}\Big] + P_0\Big[X_{\tau_1} \cdot \widehat{\ell} \ge \Big(\frac{u}{4}\Big)^{\frac{2}{7}},\ \mathcal{A}\Big] = \alpha_1 + \alpha_2\,. \tag{3.29}$$

Setting $L(u) = (\frac{u}{4})^{\frac{2}{7}}$, we have for $m \ge 1$, and large u,

$$\alpha_1 \le P[X_{T_{U_{L(u)}}} \notin \partial_+ U_{L(u)}] \le u^{-m}\,, \tag{3.30}$$

provided $0 < \lambda \le \lambda'_m(d, p)$, thanks to (3.13), (3.23). Moreover using Proposition 2.5 and Chebyshev's inequality we find

$$\alpha_2 \le \Big(\frac{u}{4}\Big)^{-\frac{2}{7}m} E_0[(X_{\tau_1} \cdot \widehat{\ell})^m, \mathcal{A}]$$

$$\le \Big(\frac{u}{4}\Big)^{-\frac{2}{7}m} \overline{E}[(\Sigma_J)^m] \le \Big(\frac{u}{4}\Big)^{-\frac{2}{7}m} \overline{E}[J^m]\, \overline{E}[(\overline{M}_1 + 4\overline{H}_1 + 2)^m]$$

(using the independence of J and $\overline{M}_i, \overline{H}_i, i \ge 1$ under $\overline{P}$),

$$\le c(\lambda, \widehat{\ell}, m)\, u^{-\frac{2}{7}m} (1 + \overline{E}[\overline{H}_1^m] + E^{\widetilde{Q}}[M^m])\,. \tag{3.31}$$

From (2.28), $\overline{E}[\overline{H}_1^m]$ is obviously finite, and for $\lambda < \lambda'_{m+d}$ and large k, we obtain

$$\widetilde{Q}[2^k \le M < 2^{k+1}] \le c(\lambda)\big(P[X_{T_{U_{2^k}}} \notin \partial_+ U_{2^k}] + P\big[X_{T_{U_{2^k}}} \in \partial_+ U_{2^k}, \widetilde{T}_0 \circ \theta_{T_{U_{2^k}}} < T_{2^{k+1}} \circ \theta_{T_{U_{2^k}}}\big]\big),$$

so that summing over all positions of $X_{T_{U_{2^k}}}$, and using translation invariance

$$\le c(\lambda)(1 + |\partial_+ U_{2^k}|)\, P[X_{T_{U_{2^k}}} \notin \partial_+ U_{2^k}] \le c(\lambda)\, 2^{\frac{7}{2}(d-1)k}\, 2^{-(m+d)\frac{7}{2}k}$$

$$\le c(\lambda)\, 2^{-(m+1)k}\,,\ \text{ thanks to (3.30) with } u = 4 \cdot 2^{\frac{7}{2}k}\,. \tag{3.32}$$

Hence $E^{\widetilde{Q}}[M^m] < \infty$, and coming back to (3.29), (3.30), (3.31), we see that for $\lambda \le \lambda'_{7m+d}$,

$$P_0[\sup_{n \le \tau_1} |X_n| \ge u, \mathcal{A}] \le u^{-m}\,,\ \text{ when } u \text{ is large}\,. \tag{3.33}$$

Thus for $m \geq 1$, $0 < \lambda \leq \lambda''_m(d, p)$, for t a large integer, and $u = t^{\frac{1}{7}}$:

$$\begin{aligned}
P_0[\tau_1 > t, \mathcal{A}] &\leq P_0[\sup_{n \leq \tau_1} |X_n| \geq u, \mathcal{A}] + P_0[T_{U_u} > t, \mathcal{I}] \\
&\overset{(3.33),(3.19)}{\leq} u^{-m} + \mathbb{P}[\Lambda_\omega(U_u) \leq u^{-5}] + c\, u^{\frac{7d}{4}}\, e^{\lambda u - t u^{-5}} \\
&\leq 3u^{-m} = 3t^{-\frac{m}{7}}\,, \qquad (3.34)
\end{aligned}$$

with the help of (3.23) in the last step. Our claim (3.27) now follows straightforwardly. □

We are now ready to prove the main result of this section pertaining to the ballistic nature of the walk when the anisotropy is weak.

Theorem 3.4. *For* $0 < \lambda \leq \lambda_2$, *(cf. Proposition* 3.3*),*

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s., } \lim_n \frac{X_n}{n} = v, \text{ with} \qquad (3.35)$$

$$v = \frac{E^Q[X_{\tau_1}]}{E^Q[\tau_1]}, \text{ so that } v \cdot \ell > 0, \text{ (see (2.20) for the definition of } Q). \qquad (3.36)$$

Moreover under P,

the $D(\mathbb{R}_+, \mathbb{R}^d)$*-valued processes* $B^n_{\cdot} = \frac{1}{\sqrt{n}}(X_{[\cdot n]} - [\cdot n]v)$ *converge in law towards a Brownian motion with non-degenerate covariance matrix* (3.37)

$$A = \frac{E^Q[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t]}{E^Q[\tau_1^2]}\,. \qquad (3.38)$$

Proof. Note that (3.35) is equivalent to

$$P\text{-a.s., } \frac{X_n}{n} \to v, \text{ as } n \to \infty, \text{ with } v \text{ in (3.36)}. \qquad (3.39)$$

With the renewal property stated in Proposition 2.3, the proofs of (3.39) and (3.37) are merely repetitions of the proofs of Proposition 2.1 of Sznitman-Zerner [28] and Theorem 4.1 of Sznitman [26], once we know that

$$E^Q[\tau_1^2] < \infty\,. \qquad (3.40)$$

However (3.40) is ensured by choosing $\lambda \leq \lambda_2$ in view of Proposition 3.3. There remains to prove the non-degeneracy of A. The proof uses a rather similar argument as in Theorem 4.1 of [26]. Namely it suffices to show that only $w = 0$ satisfies:

$$Q[w \cdot (X_{\tau_1} - \tau_1 v) = 0] = 1\,. \qquad (3.41)$$

But if $Q[2 < S_1, X_{S_1} = x] > 0$, it is straightforward to prove with similar arguments as in Lemma 2.2 and possibly modifying the path by inserting several back and forth crossings, just after time 2, of the edge the path crosses at the second step, that $Q[X_{\tau_1} = x, \tau_1 = n] > 0$ for an unbounded set of integers n. Thus (3.41) implies that $w \cdot v = 0$ and $w \cdot x = 0$ for all x with $Q[X_{S_1} = x, S_1 > 2] > 0$. Taking limits over such sites, one obtains that $w \cdot y = 0$, for any $y \in \mathbb{R}^d$ orthogonal to ℓ. Since $w \cdot v = 0$ and $v \cdot \ell > 0$, this implies $w = 0$. This finishes the proof of Theorem 3.4. □

4. Strong Anisotropy and Sub-Diffusive Behavior

We will now study the asymptotic behavior of the walk when λ is large. We will see that unlike the small λ regime where the walk on the infinite cluster has non-degenerate velocity, the large λ regime leads to a drastic slowdown. This effect is related to the presence of certain long but finite arms in the infinite cluster roughly pointing in the direction $\widehat{\ell}$, that are powerful traps when λ is large. The flavor of the results presented here is similar to Bramson-Durrett [7], see also Bramson [6].

Theorem 4.1. *There exists $\lambda_s(d, p) \geq 1$, such that for $\lambda > \lambda_s$,*

$$P_0\text{-a.s.}, \lim_n \frac{|X_n|}{n^{\lambda_s/\lambda}} = 0\,, \tag{4.1}$$

and consequently

$$\mathbb{P}\text{-a.s., for all } x \in \mathcal{C},\ P_{x,\omega}\text{-a.s.}, \lim_n \frac{|X_n|}{n^{\lambda_s/\lambda}} = 0\,. \tag{4.2}$$

Proof. Note that (4.2) is an immediate consequence of (4.1), (it is in fact equivalent), and we only need to prove (4.1). Using the notation (1.18), we define for $L > 1$:

$$\Gamma_L = \{z \in \mathbb{Z}^d, \sum_1^d |z \cdot f_i| \leq L\}\,. \tag{4.3}$$

Without loss of generality we assume from now on that $\lambda \geq 1$. Given $L_0 > 1$, and a sequence $(\delta_k)_{k\geq 0}$ in $[1, \infty)$, we are going to construct an increasing sequence $(L_k)_{k\geq 0}$, such that for $k \geq 0$:

$$L_{k+1} = L_0 + c \sum_{i=0}^k \delta_i\,, \tag{4.4}$$

$$\Gamma_{L_k} \cup \partial\Gamma_{L_k} \subseteq \Gamma_{L_{k+1}}\,, \tag{4.5}$$

for any $x \in \partial\Gamma_{L_k}$, there is a self-avoiding nearest neighbor path (4.6)
$(x_i)_{0\leq i\leq K}$, with $K \geq 2$, $x_0 = x$, $x_i \in \Gamma_{L_{k+1}} \backslash (\Gamma_{L_k} \cup \partial\Gamma_{L_k})$, $i \geq 1$, and

i) $K \leq c\,\delta_k$,

ii) $(x_K - x_0) \cdot \widehat{\ell} \geq \delta_k$,

iii) $\displaystyle\sum_{i=0}^K e^{2\ell\cdot(x_0 - x_i)} \leq c(\lambda)$,

(we refer to the end of the introduction for the convention concerning constants).

Indeed consider $L > 1$ and $\delta \geq 1$. For any $x \in \partial\Gamma_L$, we can find

$$n_x = \sum_1^d \epsilon_i f_i, \quad \text{with } \epsilon_i \in \{-1, 1\}^d\,, \tag{4.7}$$

such that the half-space $\{z \in \mathbb{Z}^d, n_x \cdot z \le n_x \cdot x\}$ contains Γ_L, and since n_x can only take finitely many values and does not belong to $\mathbb{R}_- f_1$, we can choose $v_x = v(n_x) \in S^{d-1}$, with

$$v_x \cdot n_x > 0, \text{ and } v_x \cdot f_1 \ge c_0(d) > 0, (\text{recall } f_1 = \widehat{\ell}). \tag{4.8}$$

For some $i_0 \ge 1$, depending on d, we can then construct a nearest neighbor path $\overline{x}_0 = x$, $\overline{x}_1, \dots, \overline{x}_{i_0}$, with $\overline{x}_i, i \ge 1$, outside $\Gamma_L \cup \partial\Gamma_L$ and $\overline{x}_{i_0}$ at least at distance $2+\sqrt{d}$ from the above mentioned half-space. We then pick a nearest neighbor path in the set of vertices of the cubes $y + [0,1]^d$, $y \in \mathbb{Z}^d$ which intersect the segment $[\overline{x}_{i_0}, \overline{x}_{i_0} + v_x(\delta + \sqrt{d} + 2i_0)c_0^{-1}] \subseteq \mathbb{R}^d$, starting at $\overline{x}_{i_0}$ and ending in a cube containing $\overline{x}_{i_0} + v_x(\delta + \sqrt{d} + 2i_0)c_0^{-1}$. Note that all vertices of this cube lie at distance at least $\delta + i_0$ from x. Extracting a self-avoiding path $(x_i)_{0 \le i \le K}$, from the concatenation of the above two paths we have $2 \le K \le c\,\delta$, $x_0 = x$, $x_i \notin \Gamma_L \cup \partial\Gamma_L$, for $i \ge 1$. Moreover we see that $(x_K - x_0) \cdot \widehat{\ell} \ge \delta$, and $\sum_{i=0}^K \exp\{-2\ell \cdot (x_i - x_0)\} \le c(\lambda)$.

We now see that for a suitable $c > 0$, for any L_0, $(\delta_k)_{k \ge 0}$ as above, setting $L_{k+1} = L_k + c\,\delta_k$, $k \ge 0$, we can realize (4.4), (4.5), (4.6).

We thus consider $L_0 > 1$, $(\delta_k)_{k \ge 0}$ a sequence in $[1, \infty)$, which will later be specified in (4.23), and the corresponding sequence $(L_k)_{k \ge 0}$. We define for $k \ge 0$,

$$\Delta_k = \Gamma_{L_{k+1}} \backslash (\Gamma_{L_k} \cup \partial\Gamma_{L_k}), \tag{4.9}$$

$$N_k = T_{\Gamma_{L_k}}, \text{ the exit time from } \Gamma_{L_k}, \tag{4.10}$$

and for each $x \in \partial\Gamma_{L_k}$, with $(x_i)_{0 \le i \le K}$ as in (4.6), we consider the event:

$$\begin{aligned} \mathcal{J}_x = \{\omega : \omega(\{x_i, x_{i+1}\}) = 1,\ 0 \le i < K, \text{ and } \omega(b) = 0, \text{ for any} \\ b = \{x_i, z\}, \text{ with } i \ge 1, \text{ and } z \ne x_{i-1}, x_{i+1}\}. \end{aligned} \tag{4.11}$$

Note that from (4.6) follows that for $k \ge 0$, and $x \in \partial\Gamma_{L_k}$,

$$\mathcal{J}_x \text{ is } \sigma(\omega(b), b \notin E_k)\text{-measurable}, \tag{4.12}$$

where E_k stands for the set of nearest neighbor-edges with at least one end-point in Γ_{L_k}. We will use

Lemma 4.2. *($\lambda \ge 1$). For $k \ge 0$, $n \ge 1$, P_0-a.s.,*

$$P_0[N_{k+1} \ge N_k + n \mid \mathcal{F}_{N_k}] \ge c(\lambda)(1 - c(\lambda)\, e^{-2\lambda\delta_k})_+^n\, (p(1-p))^{c\delta_k}. \tag{4.13}$$

Proof. We first prove that P_0-a.s., on $\{N_k < \infty\}$, the left-hand-side of (4.13) is bigger than

$$c(\lambda) \inf_{x \in \partial\Gamma_{L_k}} P_{x_1}[T_{\Delta_k} \ge n, \mathcal{J}_x], \text{ (with the notations of (4.6))}. \tag{4.14}$$

Indeed if h is a bounded non-negative $\mathcal{F}_{N_k}$-measurable function vanishing on $\{N_k = \infty\}$,

$$\begin{aligned} &E_0[N_{k+1} \ge N_k + n, h] \\ &= \sum_{x \in \partial\Gamma_{L_k}} \mathbb{E}[E_{0,\omega}[h, X_{N_k} = x]\, P_{x,\omega}[N_{k+1} \ge n]] \\ &\ge \sum_{x \in \partial\Gamma_{L_k}} \mathbb{E}\big[E_{0,\omega}[h, X_{N_k} = x], \mathcal{J}_x, P_{x,\omega}[X_1 = x_1, T_{\Delta_k} \circ \theta_1 \ge n]\big] \\ &\ge c(\lambda) \sum_{x \in \partial\Gamma_{L_k}} \mathbb{E}\big[E_{0,\omega}[h, X_{N_k} = x],\ \mathcal{J}_x,\ P_{x_1,\omega}[T_{\Delta_k} \ge n]\big]. \end{aligned} \tag{4.15}$$

Observe that $E_{0,\omega}[h, X_{N_k} = x]$ is $\sigma(\omega(b), b \in E_k)$-measurable, whereas the remaining terms inside the $\mathbb{P}$-expectation are $\sigma(\omega(b), b \notin E_k)$-measurable. Hence the left-hand-side of (4.15) is bigger than

$$c(\lambda)\, E_0[h] \inf_{x \in \partial\Gamma_{L_k}} P_{x_1}[T_{\Delta_k} \geq n, \mathcal{J}_x]\,,$$

and the claim in (4.14) follows. Since (4.13) is obvious on $\{N_k = \infty\}$, we only need to provide a suitable lower bound for (4.14). Note that for $x \in \partial\Gamma_{L_k}$, on $\mathcal{J}_x$, x_i, $1 \leq i \leq K$, is in essence a one-dimensional segment in Δ_k, which the walk can only exit through $x_0 = x$. Hence with the notations of (1.7), it follows that

$$P_{x_1}[T_{\Delta_k} \geq n, \mathcal{J}_x] \geq \mathbb{E}[\mathcal{J}_x, P_{x_1,\omega}(H_{x_K} < H_{x_0})\, P_{x_K,\omega}(\widetilde{H}_{x_K} < H_{x_0})^n]\,. \tag{4.16}$$

Then using for instance Chung [9], Chapter I §12, we have on $\mathcal{J}_x$

$$P_{x_1,\omega}[H_{x_K} < H_{x_0}] = \frac{\gamma_0 - \gamma_1}{\gamma_0 - \gamma_K} = (\gamma_0 - \gamma_K)^{-1}, \tag{4.17}$$

$$P_{x_K,\omega}[\widetilde{H}_{x_K} < H_{x_0}] = 1 - \frac{\gamma_{K-1} - \gamma_K}{\gamma_0 - \gamma_K}\,, \text{ where} \tag{4.18}$$

$$\gamma_0 = 0,\ \gamma_i = -\sum_{0 \leq j < i} \pi_{0,j}, \text{ for } 0 < i \leq K, \text{ with } \pi_{0,0} = 1, \text{ and} \tag{4.19}$$

$$\begin{aligned} \pi_{0,j} &= \frac{r_\omega(x_1, x_0)}{r_\omega(x_1, x_2)} \cdots \frac{r_\omega(x_j, x_{j-1})}{r_\omega(x_j, x_{j+1})}\,,\ 1 \leq j < K\,, \\ &\overset{(1.8)}{=} \frac{w_\omega(\{x_0, x_1\})}{w_\omega(\{x_j, x_{j+1}\})} = \exp\{\ell \cdot (x_0 + x_1 - x_j - x_{j+1})\},\ \ (\text{recall } \omega \in \mathcal{J}_x)\,. \end{aligned} \tag{4.20}$$

Note that from (4.6) ii) and iii), (and recall $x_0 = x \in \partial\Gamma_{L_k}$),

$$1 = \gamma_0 - \gamma_1 \leq \gamma_0 - \gamma_K \leq c(\lambda), \text{ and } \gamma_{K-1} - \gamma_K = \pi_{0,K-1} \leq c(\lambda)\, e^{-2\lambda\delta_k}\,. \tag{4.21}$$

Coming back to (4.16) we thus find

$$P_{x_1}[T_{\Delta_k} \geq n, \mathcal{J}_x] \geq c(\lambda)(1 - c(\lambda)\, e^{-2\lambda\delta_k})_+^n\, \mathbb{P}[\mathcal{J}_x], \tag{4.22}$$

and using (4.6) i) and (4.11), our claim (4.13) follows. □

We now specify L_0 and the sequence $(\delta_k)_{k \geq 0}$, via

$$L_0 = 2,\ \delta_k = c_1 \log(k \vee 1) + 1,\ k \geq 0\,, \tag{4.23}$$

where c_1 is a small enough constant which only depends on d and p, so that with the help of (4.13), we find that for large k,

$$P_0[N_{k+1} \geq N_k + e^{2\lambda\delta_k} \,|\, \mathcal{F}_{N_k}] \overset{P_0\text{-a.s.}}{\geq} k^{-\frac{1}{2}},\ \text{ and} \tag{4.24}$$

$$L_k \leq c\, k \log k\,. \tag{4.25}$$

It follows that for large k,

$$P_0[N_{i+1} < N_i + e^{2\lambda\delta[\frac{k}{2}]},\ i = \left[\frac{k}{2}\right], \dots, k-1] \leq \left(1 - k^{-\frac{1}{2}}\right)^{\frac{k}{2}} \leq e^{-\frac{\sqrt{k}}{2}}, \tag{4.26}$$

and from Borel-Cantelli's lemma, that

$$P_0\text{-a.s., for large } k,\ N_k \geq e^{2\lambda\delta[\frac{k}{2}]} \geq c(\lambda)\, k^{2c_1\lambda}. \tag{4.27}$$

Taking (4.25) into account, it follows that

$$P_0\text{-a.s.,} \lim_n \frac{|X_n|}{n^{\frac{1}{c_1\lambda}}} = 0\,. \tag{4.28}$$

Picking $\lambda_s = \frac{1}{c_1}$, the claim (4.1) follows. □

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